Nonplanar Periodic Solutions for Spatial Restricted 3-Body and 4-Body Problems *

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Abstract: In this paper, we study the existence of non-planar periodic solutions for the following spatial restricted 3-body and 4-body problems: for $N = 2$ or 3, given any positive masses $m_1, \cdots, m_N$, the mass points of $m_1, \cdots, m_N$ move in the plane of N circular obits centered at the center of masses, the sufficiently small mass moves on the perpendicular axis passing the center of masses. Using variational minimizing methods, we establish the existence of the minimizers of the Lagrangian action on anti-T/2 or odd symmetric loop spaces. Moreover, we prove these minimizers are non-planar periodic solutions by using the Jacobi's Necessary Condition for local minimizers.

Keywords: Restricted 3-body problems; Restricted 4-body problems; nonplanar periodic solutions; variational minimizers; Jacobi's Necessary Conditions.

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1 Introduction and Main Results

In this paper, we study the spatial circular restricted 3-body and 4-body problems. For $N = 2$ or 3, suppose points of positive masses $m_1, \cdots, m_N$ move in the plane of their circular orbits $q_1(t), \cdots, q_N(t)$ with the radius $r_1, \cdots, r_N > 0$ and the center of masses is at the origin; suppose the sufficiently small mass point does not influence the motion of $m_1, \cdots, m_N$, and moves on the vertical axis of the moving plane for the given masses $m_1, \cdots, m_N$, here the vertical axis passes through the center of masses.

It is known that $q_1(t), \cdots, q_N(t) (N = 2 \text{ or } 3)$ satisfy the Newtonian equations:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, \cdots, N, \tag{1.1}$$

where

$$U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|q_i - q_j|}. \tag{1.2}$$

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for sufficiently small mass is governed by the gravitational forces of $m_1, \cdots, m_N (N = 2 \text{ or } 3)$ and therefore it satisfies the following equation

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i (q_i - q)}{|q_i - q|^3}, \ N = 2 \text{ or } 3. \tag{1.3}$$

For $N \ge 2$, there are many papers concerned with the restricted N-body problem, see [3,4,6,8-10] and the references therein. In [8], Sitnikov considered the following model: two mass points of equal mass $m_1 = m_2 = \frac{1}{2}$ move in the plane of their elliptic orbits and the center of masses is

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at rest, the third mass point which does not influence the motion of the first two moves on the line perpendicular to the plane containing the first two mass points and goes through the center of mass, and he used geometrical methods to prove the existence of the oscillatory parabolic orbit of

$$\ddot{z}(t) = \frac{-z(t)}{(|z(t)|^2 + |r(t)|^2)^{3/2}}, \tag{1.4}$$

where $r(t) = r(t+2\pi) > 0$ is the distance from the center of mass to one of the first two mass points. McGehee [6] used the stable and unstable manifolds to study the homoclinic orbits (parabolic orbits) of (1.4). In[4], Mathlouthi studied the periodic solutions for the spatial circular restricted 3-body problems by minimax variational methods. Recently, Li, Zhang and Zhao[3] used variational minimizing methods to study spatial circular restricted N+1-body problem with a zero mass moving on the vertical axis of the moving plane for N equal mass.

Motivated by [3], we use the Jacobi's Necessary Condition for local minimizers to further study the spatial circular restricted 3-body and 4-body problems with a sufficiently small mass moving on the perpendicular axis of the circular orbits plane for any given masses $m_1, \cdots, m_N (N = 2$ or $3)$.

Define

$$W^{1,2}(R/TZ, R) = \left\{ u(t) \middle| u(t), u'(t) \in L^2(R, R),\ u(t+T) = u(t) \right\}.$$

The inner product and the norm of $W^{1,2}(R/TZ, R)$ are

$$< u, v >= \int_0^T (uv + u' \cdot v')dt, \tag{1.5}$$

$$\|u\| = \Big[\int_0^T |u|^2 dt\Big]^{\frac{1}{2}} + \Big[\int_0^T |u'|^2 dt\Big]^{\frac{1}{2}}. \tag{1.6}$$

The functional corresponding to the equation (1.3) is

$$\begin{aligned} f(q) &= \int_0^T \Big[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^N \frac{m_i}{|q - q_i|}\Big] dt, \qquad q \in \Lambda_j, j = 1, 2 \\ &= \int_0^T \Big[\frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \cdots + \frac{m_N}{\sqrt{r_N^2 + z^2}}\Big] dt \triangleq f(z),\ N = 2 \text{ or } 3, \end{aligned} \tag{1.7}$$

where

$$\Lambda_1 = \left\{ q(t) = (0, 0, z(t)) \middle| z(t) \in W^{1,2}(R/TZ, R),\ z(t + \frac{T}{2}) = -z(t) \right\},$$

and

$$\Lambda_2 = \left\{ q(t) = (0, 0, z(t)) \middle| z(t) \in W^{1,2}(R/TZ, R),\ z(-t) = -z(t) \right\}.$$

Our main results are the following:

Theorem 1.1 For $N = 2$, the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar and noncollision periodic solution.

Theorem 1.2 For $N = 3$, the minimizer of $f(q)$ on the closure $\overline{\Lambda}_i$ of $\Lambda_i (i = 1, 2)$ is a nonplanar and noncollision periodic solution.

2 Preliminaries

In this section, we will list some basic Lemmas and inequality for proving our Theorems 1.1 and 1.2.

Lemma 2.1(Palais's Symmetry Principle([7])) Let σ be an orthogonal representation of a finite or compact group G, H be a real Hilbert space, $f: H \to R$ satisfies $f(\sigma \cdot x) = f(x), \forall \sigma \in G, \forall x \in H$.

Set $F = \{x \in H | \sigma \cdot x = x, \ \forall \sigma \in G\}$. Then the critical point of f in F is also a critical point of f in H.

Remark 2.1 By Palais's Symmetry Principle, we know that the critical point of $f(q)$ in $\overline{\Lambda}_i = \Lambda_i (i = 1, 2)$ is a periodic solution of Newtonian equation (1.3).

Lemma 2.2(Tonelli[1]) Let X be a reflexive Banach space, S be a weakly closed subset of X, $f: S \to R \cup +\infty$. If $f \not\equiv +\infty$ is weakly lower semi-continuous and coercive($f(x) \to +\infty$ as $\|x\| \to +\infty$), then f attains its infimum on S.

Lemma 2.3(Poincare-Wirtinger Inequality[5]) Let $q \in W^{1,2}(R/TZ, R^K)$ and $\int_0^T q(t)dt = 0$, then

$$\int_0^T |q(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{q}(t)|^2 dt. \tag{2.1}$$

Lemma 2.4 $f(q)$ in (1.7) attains its infimum on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$.

Proof. By using Lemma 2.3, for $\forall z \in \Lambda_i, \ i = 1, 2$, the equivalent norm of (1.6) in $\Lambda_i (i = 1, 2)$ is

$$\|z\| \cong \Big[\int_0^T |z'|^2 dt \Big]^{\frac{1}{2}}. \tag{2.2}$$

Hence by the definitions of $f(q)$, it is easy to see that f is C^1 and coercive on $\Lambda_i (i = 1, 2)$. In order to get Lemma 2.4, we only need to prove that f is weakly lower semi-continuous on $\Lambda_i (i = 1, 2)$. In fact, for $\forall z_n \in \Lambda_i$, if $z_n \rightharpoonup z$ weakly, by compact embedding theorem, we have the uniformly convergence:

$$\max_{0 \leq t \leq T} |z_n(t) - z(t)| \to 0, \qquad n \to \infty, \tag{2.3}$$

which implies

$$\int_0^T \frac{m_1}{\sqrt{r_1^2 + z_n^2}} + \cdots + \frac{m_N}{\sqrt{r_N^2 + z_n^2}} dt \to \int_0^T \frac{m_1}{\sqrt{r_1^2 + z^2}} + \cdots + \frac{m_N}{\sqrt{r_N^2 + z^2}} dt, \ N = 2 \text{ or } 3. \tag{2.4}$$

It is well-known that the norm and its square are weakly lower semi-continuous. Therefore, combined with (2.4), one has

$$\liminf_{n \to \infty} f(z_n) \geq f(z),$$

that is, f is weakly lower semi-continuous on $\Lambda_i (i = 1, 2)$. By lemma 2.2, we can get that $f(q)$ in (1.7) attains its infimum on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$. $\square$

Lemma 2.5(Jacobi's Necessary Condition[2]) Let $F \in C^3([a, b] \times R \times R, R)$. If the critical point $y = \tilde{y}(x)$ corresponds to a minimum of the functional $\int_a^b F(x, y(x), y'(x))dx$ on $M = \{y \in W^{1,2}([a, b], R) | y(a) = A, y(b) = B\}$ and if $F_{y'y'} > 0$ along this critical point, then the open interval (a, b) contains no points conjugate to a, that is, for $\forall c \in (a, b)$, the following problem:

$$\begin{cases} -\frac{d}{dx}(Ph') + Qh = 0, \\ h(a) = 0, \quad h(c) = 0, \end{cases} \tag{2.5}$$

has only the trivial solution $h(x) \equiv 0,\ \forall x \in (a,c)$, where

$$P = \frac{1}{2} F_{y'y'}|_{y=\tilde{y}}, \tag{2.6}$$

$$Q = \frac{1}{2}\Big(F_{yy} - \frac{d}{dx}F_{yy'}\Big)\Big|_{y=\tilde{y}}. \tag{2.7}$$

Remark 2.2 It is easy to see that Lemma 2.5 is suitable for the fixed end problem. In this paper, we consider the periodic solutions of (1.3) on $\overline{\Lambda}_i = \Lambda_i (i = 1, 2)$, hence we need to establish a similar conclusion as Lemma 2.5 for the periodic boundary problem.

Lemma 2.6 Let $F \in C^3(R \times R \times R, R)$. Assume that $u = \tilde{u}(t)$ is a critical point of the functional $\int_0^T F(t, u(t), u'(t))dt$ on $W^{1,2}(R/TZ, R)$ and $F_{u'u'}|_{u=\tilde{u}} > 0$. If the open interval $(0, T)$ contains a point c conjugate to 0, then $u = \tilde{u}(t)$ is not a minimum of the functional $\int_0^T F(t, u(t), u'(t))dt$.

Proof. Suppose $u = \tilde{u}(t)$ is a minimum of the functional $\int_0^T F(t, u(t), u'(t))dt$. The second variation of $\int_0^T F(t, u(t), u'(t))dt$ is

$$\int_0^T (Ph'^2 + Qh^2)dt, \tag{2.8}$$

where

$$P = \frac{1}{2} F_{u'u'}|_{u=\tilde{u}}, \tag{2.9}$$

$$Q = \frac{1}{2}\Big(F_{uu} - \frac{d}{dt}F_{uu'}\Big)\Big|_{u=\tilde{u}}. \tag{2.10}$$

Set

$$Q_{\tilde{u}}(h) = \int_0^T (Ph'^2 + Qh^2)dt. \tag{2.11}$$

For $\forall h \in C_0^1([0, T], R)$, it is easy to see that $Q_{\tilde{u}}(h) \geq 0$. Then by $Q_{\tilde{u}}(\theta) = 0$, θ is a minimum of $Q_{\tilde{u}}(h)$. The Euler-Lagrange equation which is called the Jacobi equation of (2.11) is

$$-\frac{d}{dt}(Ph') + Qh = 0. \tag{2.12}$$

Since the interval $(0, T)$ contains a point c conjugate to 0, there exists a nonzero Jacobi field $h_0 \in C^2([0, T], R)$ satisfying

$$\begin{cases} -\frac{d}{dt}(Ph_0') + Qh_0 = 0, \\ h_0(0) = 0, \quad h_0(c) = 0. \end{cases} \tag{2.13}$$

Let

$$\hat{h}(t) = \begin{cases} h_0(t) & t \in [0, c], \\ 0 & t \in (c, T], \end{cases} \tag{2.14}$$

we have $\hat{h} \in C^2([0, T]\backslash\{c\}, R)$, $\hat{h}(0) = \hat{h}(c) = \hat{h}(T) = 0$ and

$$Q_{\tilde{u}}(\hat{h}) = \int_0^T (P\hat{h}'^2 + Q\hat{h}^2)dt = \int_0^c (Ph_0'^2 + Qh_0^2)dt = 0. \tag{2.15}$$

Notice that we can extend $\hat{h}$ periodically when we take T as the period, so $\hat{h} \in W_0^{1,2}(R/TZ, R)$. For $\forall h \in C_0^1([0, T], R)$, it is easy to check that $Q_{\tilde{u}}(h) \geq 0$. Then by (2.15), one has $\hat{h} \in C^2([0, T]\backslash\{c\}, R) \cap W_0^{1,2}(R/TZ, R)$ is a minimum of $Q_{\tilde{u}}(h)$. Hence we get

$$-\frac{d}{dt}(P\hat{h}') + Q\hat{h} = 0. \tag{2.16}$$

Combine with $\hat{h}(0) = \hat{h}(c) = 0$, by the uniqueness of initial value problems for second order differential equation, we have $\hat{h}(t) \equiv 0$ on [0,c], which contradicts the definition of $\hat{h}$. Therefore, Lemma 2.6 holds. □

3 Proof of Theorem 1.1

In this section, we consider the spatial circular restricted 3-body problem with a sufficiently small mass moving on the vertical axis of the moving plane for arbitrary given positive masses m_1, m_2. Suppose the planar circular orbits are

$$q_1(t) = r_1 e^{\sqrt{-1}\frac{2\pi}{T}t}, \qquad q_2(t) = -r_2 e^{\sqrt{-1}\frac{2\pi}{T}t}, \tag{3.1}$$

here the radius r_1, r_2 are positive constants depending on $m_i (i = 1, 2)$ and T (see Lemma 3.1). We also assume that

$$m_1 q_1(t) + m_2 q_2(t) = 0. \tag{3.2}$$

The functional corresponding to the equation (1.3) is

$$\begin{aligned} f(q) &= \int_0^T \Big[\frac{1}{2}|\dot{q}|^2 + \frac{m_1}{|q - q_1|} + \frac{m_2}{|q - q_2|}\Big] dt, \qquad q \in \Lambda_i,\ i = 1, 2 \\ &= \int_0^T \Big[\frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \frac{m_2}{\sqrt{r_2^2 + z^2}}\Big] dt \triangleq f(z). \end{aligned} \tag{3.3}$$

Lemma 3.1 The radius r_1, r_2 of the planar circular orbits for the masses m_1, m_2 are

$$r_1 = \Big(\frac{T}{2\pi(m_1 + m_2)}\Big)^{\frac{2}{3}} m_2, \qquad r_2 = \Big(\frac{T}{2\pi(m_1 + m_2)}\Big)^{\frac{2}{3}} m_1.$$

Proof. Substituting (3.1) into (3.2), it is easy to get

$$r_2 = \frac{m_1}{m_2} r_1. \tag{3.4}$$

It follows from (1.1) and (1.2) that

$$\ddot{q}_1 = m_2 \frac{q_2 - q_1}{|q_2 - q_1|^3}. \tag{3.5}$$

Then by (3.1) and (3.4), we have

$$-\frac{4\pi^2}{T^2} q_1 = m_2 \frac{(-\frac{m_1}{m_2} - 1) q_1}{r_1^3 | -\frac{m_1}{m_2} - 1|^3}, \tag{3.6}$$

which implies

$$r_1 = \Big(\frac{T}{2\pi(m_1 + m_2)}\Big)^{\frac{2}{3}} m_2. \tag{3.7}$$

Hence by (3.4), one has

$$r_2 = \Big(\frac{T}{2\pi(m_1 + m_2)}\Big)^{\frac{2}{3}} m_1. \quad \square \tag{3.8}$$

Proof of Theorem 1.1 Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$. For the functional (3.3), let

$$F(z, z') = \frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \frac{m_2}{\sqrt{r_2^2 + z^2}}.$$

Then the second variation of (3.3) in the neighborhood of $z = 0$ is given by

$$\int_0^T (Ph'^2 + Qh^2) dt, \tag{3.9}$$

where

$$P = \frac{1}{2}F_{z'z'}|_{z=0} = \frac{1}{2}, \tag{3.10}$$

$$Q = \frac{1}{2}\Big(F_{zz} - \frac{d}{dt}F_{zz'}\Big)\Big|_{z=0} = -\Big(\frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3}\Big). \tag{3.11}$$

The Euler equation of (3.9) is called the Jacobi equation of the original functional (3.3), which is

$$-\frac{d}{dt}(Ph') + Qh = 0, \tag{3.12}$$

that is,

$$h'' + \Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\Big)h = 0. \tag{3.13}$$

Next, we study the solution of (3.13) with initial values $h(0) = 0,\ h'(0) = 1$. It is easy to get

$$h(t) = \sqrt{\frac{r_1^3 r_2^3}{m_2 r_1^3 + m_1 r_2^3}} \cdot sin\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}}t. \tag{3.14}$$

It follows from (3.7) and (3.8)that

$$\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}} = \sqrt{\frac{m_1^4 + m_2^4}{m_1^3 m_2^3}}(m_1 + m_2) \cdot \frac{2\pi}{T}. \tag{3.15}$$

Hence

$$h(t) = \frac{\sqrt{m_1^3 m_2^3}T}{2\pi\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} \cdot sin\left(\sqrt{\frac{m_1^4 + m_2^4}{m_1^3 m_2^3}}(m_1 + m_2) \cdot \frac{2\pi}{T}t\right), \tag{3.16}$$

which is not identically zero on $[0, \frac{\sqrt{m_1^3 m_2^3}T}{\sqrt{m_1^4+m_2^4}(m_1+m_2)}]$. Since

$$m_1^6 + m_2^6 \geq 2\sqrt{m_1^6 \cdot m_2^6} = 2m_1^3 m_2^3 > m_1^3 m_2^3, \tag{3.17}$$

one has

$$\begin{aligned}(m_1^4 + m_2^4)(m_1 + m_2)^2 &> m_1^6 + m_2^6 \\ &> m_1^3 m_2^3,\end{aligned} \tag{3.18}$$

which implies

$$\frac{\sqrt{m_1^3 m_2^3}}{\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} < 1. \tag{3.19}$$

Therefore

$$\frac{\sqrt{m_1^3 m_2^3}T}{\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} < T. \tag{3.20}$$

Choose $0 < c = \frac{\sqrt{m_1^3 m_2^3}T}{2\sqrt{m_1^4+m_2^4}(m_1+m_2)} < \frac{T}{2}$, we have

$$h(c) = \frac{\sqrt{m_1^3 m_2^3}T}{2\pi\sqrt{m_1^4 + m_2^4}(m_1 + m_2)} \cdot sin\pi = 0. \tag{3.21}$$

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

Let

$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \tag{3.22}$$

It is easy to check that $\tilde{h}(t) \in C^2([0,T]\backslash\{c, \frac{T}{2}, \frac{T}{2}+c\}, R) \cap W^{1,2}(R,R)$, $\tilde{h}(t+\frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (3.12). Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$. Then by Lemma 2.6, $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_1. Hence the minimizers of $f(q)$ on Λ_1 are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get the non-planar periodic solutions.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0,c], \\ 0 & t \in (c, T-c], \\ -h(T-t) & t \in (T-c, T]. \end{cases} \tag{3.23}$$

It is easy to check that $\bar{h}(t) \in C^2([0,T]\backslash\{c, T-c\}, R) \cap W^{1,2}(R,R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (3.12). Notice that we can extend $\bar{h}$ periodically when we take T as the period, so $\bar{h} \in \Lambda_2$. Then by Lemma 2.6, $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_2. Hence the minimizers of $f(q)$ on Λ_2 are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get the non-planar periodic solutions.

Combined with Lemma 2.4, the proof is completed. □

4 Proof of Theorem 1.2

In this section, we consider the spatial circular restricted 4-body problem with a sufficiently small mass moving on the vertical axis of the moving plane for arbitrary positive masses m_1, m_2, m_3. Suppose there exists $\theta_1, \theta_2, \theta_3 \in (0, 2\pi)$ such that the planar circular orbits are

$$q_1(t) = r_1 e^{\sqrt{-1}\frac{2\pi}{T}t} e^{\sqrt{-1}\theta_1}, \quad q_2(t) = r_2 e^{\sqrt{-1}\frac{2\pi}{T}t} e^{\sqrt{-1}\theta_2}, \quad q_3(t) = r_3 e^{\sqrt{-1}\frac{2\pi}{T}t} e^{\sqrt{-1}\theta_3}, \tag{4.1}$$

here the radius r_1, r_2, r_3 are positive constants depending on $m_i (i = 1,2,3)$ and T (see Lemma 4.2). We also assume that

$$m_1 q_1(t) + m_2 q_2(t) + m_3 q_3(t) = 0 \tag{4.2}$$

and

$$|q_i - q_j| = l, \quad 1 \le i \ne j \le 3, \tag{4.3}$$

where the constant $l > 0$ depends on $m_i (i = 1,2,3)$ and T (see Lemma 4.1). The functional corresponding to the equation (1.3) is

$$\begin{aligned} f(q) &= \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{m_1}{|q - q_1|} + \frac{m_2}{|q - q_2|} + \frac{m_3}{|q - q_3|}\right] dt, \qquad q \in \Lambda_i, \ i = 1,2 \\ &= \int_0^T \left[\frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \frac{m_2}{\sqrt{r_2^2 + z^2}} + + \frac{m_3}{\sqrt{r_3^2 + z^2}}\right] dt \triangleq f(z). \end{aligned} \tag{4.4}$$

In order to get Theorem 1.2, we firstly prove Lemmas 4.1 and 4.2 as follows.

Lemma 4.1 Let $M = m_1 + m_2 + m_3$, we have $l = \sqrt[3]{\frac{MT^2}{4\pi^2}}$.

Proof. It follows from (1.1) and (1.2) that

$$\ddot{q}_1 = m_2 \frac{q_2 - q_1}{|q_2 - q_1|^3} + m_3 \frac{q_3 - q_1}{|q_3 - q_1|^3}. \tag{4.5}$$

Then by (4.1)-(4.3), one has

$$-\frac{4\pi^2}{T^2}q_1 = \frac{1}{l^3}(m_2q_2 + m_3q_3 - m_2q_1 - m_3q_1) \\ = \frac{1}{l^3}(-m_1q_1 - m_2q_1 - m_3q_1), \tag{4.6}$$

which implies

$$l^3 = \frac{MT^2}{4\pi^2}, \tag{4.7}$$

that is,

$$l = \sqrt[3]{\frac{MT^2}{4\pi^2}}. \quad \square \tag{4.8}$$

Lemma 4.2 The radius r_1, r_2, r_3 of the planar circular orbits for the masses m_1, m_2, m_3 are

$$r_1 = \frac{\sqrt{m_2^2 + m_2m_3 + m_3^2}}{M}l,$$

$$r_2 = \frac{\sqrt{m_1^2 + m_1m_3 + m_3^2}}{M}l,$$

$$r_3 = \frac{\sqrt{m_1^2 + m_1m_2 + m_2^2}}{M}l.$$

Proof. Choose the geometrical center of the initial configuration $(q_1(0), q_2(0), q_3(0))$ as the origin of the coordinate (x,y). Without loss of generality, by (4.3), we suppose the location coordinates of $q_1(0), q_2(0), q_3(0)$ are $A_1(\frac{\sqrt{3}l}{3}, 0), A_2(-\frac{\sqrt{3}l}{6}, \frac{l}{2}), A_3(-\frac{\sqrt{3}l}{6}, -\frac{l}{2})$. Then we can get the coordinate of the center of masses m_1, m_2, m_3 is $C(\frac{\frac{\sqrt{3}}{3}m_1l - \frac{\sqrt{3}}{6}m_2l - \frac{\sqrt{3}}{6}m_3l}{M}, \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M})$. To make sure the Assumption (4.2) holds, we introduce the new coordinate

$$\begin{cases} X = x - \frac{\frac{\sqrt{3}}{3}m_1l - \frac{\sqrt{3}}{6}m_2l - \frac{\sqrt{3}}{6}m_3l}{M}, \\ Y = y - \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M}. \end{cases}$$

Hence in the new coordinate (X,Y), the location coordinates of $q_1(0), q_2(0), q_3(0)$ are $A_1(\frac{\frac{\sqrt{3}}{2}m_2l + \frac{\sqrt{3}}{2}m_3l}{M}, \frac{-\frac{m_2}{2}l + \frac{m_3}{2}l}{M})$, $A_2(-\frac{\frac{\sqrt{3}}{2}m_1l}{M}, \frac{\frac{m_1}{2}l + m_3l}{M})$, $A_3(-\frac{\frac{\sqrt{3}}{2}m_1l}{M}, -\frac{\frac{m_1}{2}l + m_2l}{M})$ and the center of masses m_1, m_2, m_3 is at the origin $O(0,0)$. Then compared with (4.1), we have

$$r_1 = |A_1O| = \frac{\sqrt{m_2^2 + m_2m_3 + m_3^2}}{M}l, \tag{4.9}$$

$$r_2 = |A_2O| = \frac{\sqrt{m_1^2 + m_1m_3 + m_3^2}}{M}l, \tag{4.10}$$

$$r_3 = |A_3O| = \frac{\sqrt{m_1^2 + m_1m_2 + m_2^2}}{M}l, \tag{4.11}$$

and

$$\tan\theta_1 = \frac{\sqrt{3}(-m_2 + m_3)}{3(m_2 + m_3)}, \quad \tan\theta_2 = -\frac{\sqrt{3}(m_1 + 2m_3)}{3m_1}, \quad \tan\theta_3 = \frac{\sqrt{3}(m_1 + 2m_2)}{3m_1}. \quad \square \tag{4.12}$$

Proof of Theorem 1.2 Clearly, $q(t) = (0,0,0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$. For the functional (4.4), let

$$F(z, z') = \frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \frac{m_2}{\sqrt{r_2^2 + z^2}} + \frac{m_3}{\sqrt{r_3^2 + z^3}}.$$

Then the second variation of (4.4) in the neighborhood of $z = 0$ is given by

$$\int_0^T (Ph'^2 + Qh^2)dt, \tag{4.13}$$

where

$$P = \frac{1}{2} F_{z'z'}|_{z=0} = 1, \tag{4.14}$$

$$Q = \frac{1}{2}\Big(F_{zz} - \frac{d}{dt}F_{zz'}\Big)\Big|_{z=0} = -\Big(\frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3} + \frac{m_3}{2r_3^3}\Big). \tag{4.15}$$

The Euler equation of (4.13) is called the Jacobi equation of the original functional (4.4), which is

$$-\frac{d}{dt}(Ph') + Qh = 0, \tag{4.16}$$

that is,

$$h'' + \Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\Big)h = 0. \tag{4.17}$$

Next, we study the solution of (4.17) with initial values $h(0) = 0,\ h'(0) = 1$. It is easy to get

$$h(t) = \sqrt{\frac{r_1^3 r_2^3 r_3^3}{m_3 r_1^3 r_2^3 + m_2 r_1^3 r_3^3 + m_1 r_2^3 r_3^3}} \cdot sin\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}}t. \tag{4.18}$$

Let

$$A = \frac{\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{M},$$
$$B = \frac{\sqrt{m_1^2 + m_1 m_3 + m_3^2}}{M},$$
$$C = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{M}.$$

It follows from (4.8)-(4.11) that

$$\begin{aligned}\sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}} &= \sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}\sqrt{\frac{1}{l^3}} \\ &= \sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}} \cdot \frac{2\pi}{\sqrt{M}T}.\end{aligned} \tag{4.19}$$

Hence

$$h(t) = \frac{\sqrt{M}T}{2\pi\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}} \cdot sin\Bigg(\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}} \cdot \frac{2\pi}{\sqrt{M}T}t\Bigg), \tag{4.20}$$

which is not identically zero on $[0, \frac{\sqrt{M}T}{\sqrt{\frac{m_1}{A^3}+\frac{m_2}{B^3}+\frac{m_3}{C^3}}}]$. It is easy to check that

$$\begin{aligned}M^2 &> m_1^2 + m_1 m_2 + m_2^2, \\ M^2 &> m_1^2 + m_1 m_3 + m_3^2, \\ M^2 &> m_2^2 + m_2 m_3 + m_3^2,\end{aligned} \tag{4.21}$$

which implies

$$\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3} > m_1 + m_2 + m_3 = M. \tag{4.22}$$

Therefore

$$\frac{\sqrt{M}T}{\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}} < T. \tag{4.23}$$

Choose $0 < c = \frac{\sqrt{M}T}{2\sqrt{\frac{m_1}{A^3}+\frac{m_2}{B^3}+\frac{m_3}{C^3}}} < \frac{T}{2}$, we have

$$h(c) = \frac{\sqrt{M}T}{2\pi\sqrt{\frac{m_1}{A^3}+\frac{m_2}{B^3}+\frac{m_3}{C^3}}} \cdot sin\pi = 0. \tag{4.24}$$

Case 1: Minimizing $f(q)$ on $\bar{\Lambda}_1 = \Lambda_1$.

Let

$$\tilde{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, \frac{T}{2}], \\ -h(t - \frac{T}{2}) & t \in (\frac{T}{2}, \frac{T}{2} + c], \\ 0 & t \in (\frac{T}{2} + c, T]. \end{cases} \tag{4.25}$$

It is easy to check that $\tilde{h}(t) \in C^2([0,T]\backslash\{c, \frac{T}{2}, \frac{T}{2}+c\}, R) \cap W^{1,2}(R,R)$, $\tilde{h}(t+\frac{T}{2}) = -\tilde{h}(t)$, $\tilde{h}(0) = h(0) = 0$, $\tilde{h}(c) = h(c) = 0$ and $\tilde{h}$ is a nonzero solution of (4.16). Notice that we can extend $\tilde{h}$ periodically when we take T as the period, so $\tilde{h} \in \Lambda_1$. Then by Lemma 2.6, $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_1. Hence the minimizers of $f(q)$ on Λ_1 are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get the non-planar periodic solutions.

Case 2: Minimizing $f(q)$ on $\bar{\Lambda}_2 = \Lambda_2$.

Let

$$\bar{h}(t) = \begin{cases} h(t) & t \in [0, c], \\ 0 & t \in (c, T-c], \\ -h(T-t) & t \in (T-c, T]. \end{cases} \tag{4.26}$$

It is easy to check that $\bar{h}(t) \in C^2([0,T]\backslash\{c, T-c\}, R) \cap W^{1,2}(R,R)$, $\bar{h}(-t) = -\bar{h}(t)$, $\bar{h}(0) = h(0) = 0$, $\bar{h}(c) = h(c) = 0$ and $\bar{h}$ is a nonzero solution of (4.16). Notice that we can extend $\bar{h}$ periodically when we take T as the period, so $\bar{h} \in \Lambda_2$. Then by Lemma 2.6, $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_2. Hence the minimizers of $f(q)$ on Λ_2 are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get the non-planar periodic solutions.

Combined with Lemma 2.4, the proof is completed. □

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